Jackson

7.1 (a)

Linear polarization basis

$s_0 + s_1 = 2a_1^2$ $\qquad$ $s_0 - s_1 = 2a_2^2$

$a_1 = 1$, $a_2 = \sqrt{2}$.

$$\delta_2 - \delta_1 = \cos^{-1}\left[\frac{s_2}{2a_1 a_2}\right] = \cos^{-1}\left[\frac{1}{\sqrt{2}}\right] = \frac{\pi}{4}$$

WLOG, let $\delta_1 = 0$, $\delta_2 = \frac{\pi}{4}$

Then $E_1 = a_1 = \boxed{1}$ $\qquad$ $E_2 = \boxed{\sqrt{2}\, e^{i\pi/4}}$

$$\vec{E} = \mathrm{Re}\left[(E_1 \hat{x} + E_2 \hat{y})\, e^{i\vec{k}\cdot\vec{x} - i\omega t}\right]$$

$$\simeq \mathrm{Re}\left[(\hat{x} + \sqrt{2}\, e^{i\pi/4}\hat{y})\, e^{-i\omega t}\right] \quad \text{for fixed } \vec{x}.$$

$$\Rightarrow \quad E_x = \cos(-\omega t), \quad E_y = \sqrt{2}\cos(-\omega t + \tfrac{\pi}{4})$$

To find how much the principle axes of the ellipse is rotated, solve for when

$$|\vec{E}|^2 = \cos^2(-\omega t) + 2\cos^2(-\omega t + \tfrac{\pi}{4})$$

is an extremum.

~~d~~ Letting $\theta = \omega t$, then $|E|^2(\theta) = \cos^2(-\theta) + 2\cos^2(-\theta + \frac{\pi}{4})$

$$\frac{d|E|^2}{d\theta} = 2\cos(-\theta)(-\sin(-\theta))(-1) + 4\cos(-\theta + \frac{\pi}{4})\sin(-\theta + \frac{\pi}{4})$$

$$= 2\cos(-\theta)\sin(-\theta) + 4\cos(-\theta + \frac{\pi}{4})\sin(-\theta + \frac{\pi}{4}) = 0.$$

$$\Rightarrow \quad \cos(-\theta)\sin(-\theta) = -2\cos(-\theta + \frac{\pi}{4})\sin(-\theta + \frac{\pi}{4})$$

$$\frac{1}{2}\sin(-2\theta) = -\sin(-2\theta + \frac{\pi}{2})$$

$$\frac{1}{2}\sin(-2\theta) = -\cos(-2\theta)$$

$$\tan(-2\theta) = -2$$

$$\theta = -\frac{1}{2}\tan^{-1}(-2)$$

$$= 0.553 \approx \boxed{31.7^\circ}$$

$$|E|^2\Big|_{\theta = 31.7^\circ} = \cos^2(-0.553) + 2\cos^2(-0.553 + \frac{\pi}{4})$$

$$\approx 0.7241 + 1.8939 = 2.618$$

$\Rightarrow$ The principal axis is $\sqrt{2.618} = \boxed{1.618,}$ and $\sqrt{3-2.618} = \boxed{0.618}$

$\Rightarrow$ ~~the~~

~~etten~~ elliptical polarization basis.

$$S_0 + S_3 = 2a_+^2, \qquad S_0 - S_3 = 2a_-^2$$

$$\Rightarrow \quad a_+ = \frac{1}{\sqrt{2}} \qquad a_- = \sqrt{\frac{5}{2}}$$

$$\delta_- - \delta_+ = \cos^{-1}\left[\frac{S_1}{2a_+a_-}\right]$$

$$= \cos^{-1}\left[\frac{-1}{2\frac{\sqrt{5}}{2}}\right]$$

$$= \cos^{-1}\left[-\frac{1}{\sqrt{5}}\right] \simeq 2.03 \equiv \alpha$$

WLOG, let $\delta_+ = 0$, then $\delta_- = \alpha = 2.03$

$$\Rightarrow \boxed{E_+ = \frac{1}{\sqrt{2}}}, \quad \boxed{E_- = \sqrt{\frac{5}{2}}\, e^{i\alpha}}$$

Ratio of semimajor axes given by $\left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right| \simeq 2.618$

$$a^2+b^2 = 3, \quad \left(a^2 + \frac{a^2b^2}{a^2}\right) = 3$$

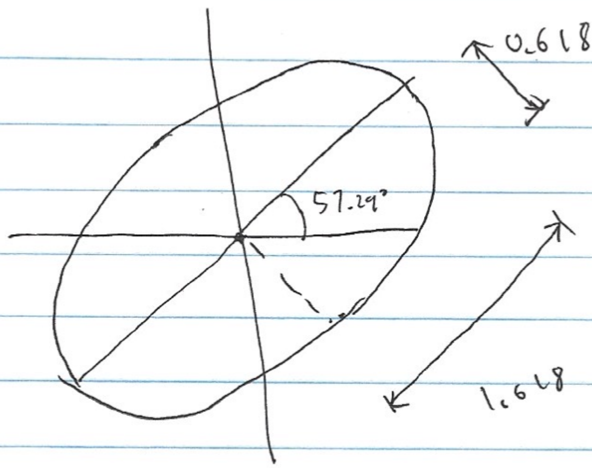
$$a^2\left[1+\left(\frac{b}{a}\right)^2\right] = 3$$

$$a^2 = 3/\left[1+(2.618)^2\right]$$

$$\Rightarrow \boxed{\begin{aligned} a &= 0.618 \\ b &= 1.618 \end{aligned}}$$

The major axis has length 1.618, minor axis has length 0.618.

The phase difference is $\alpha \approx 2.03$, thus the ellipse is rotated by $\alpha/2 \approx 1.01 \approx 57.29°$



Jackson 7.1 (b)

Linear polarization basis.

$a_1 = \sqrt{\frac{25}{2}}$, $a_2 = \sqrt{\frac{25}{2}}$

$\Rightarrow$ The polarization has equal max amplitude in x, y.

$$\delta_2 - \delta_1 = \cos^{-1}\left[\frac{s_2}{2a_1a_2}\right] = \cos^{-1}\left[\frac{24}{25}\right] \approx 0.28$$

$$\equiv \alpha \approx 16^\circ$$

WLOG, $\delta_1 = 0$, $\delta_2 = \alpha$, then

$$\boxed{E_1 = \sqrt{\frac{25}{2}}} \quad \boxed{E_2 = \sqrt{\frac{25}{2}}\, e^{i\alpha}}$$

$$\vec{E}(\text{fixed } \vec{x}, t) = \mathrm{Re}\sqrt{\frac{25}{2}}\left(\hat{x} + e^{i\alpha}\hat{y}\right)e^{-i\omega t}$$

$$= \sqrt{\frac{25}{2}}\left[\cos(-\omega t)\,\hat{x} + \cos(-\omega t + \alpha)\,\hat{y}\right]$$

Solve for $|E|^2$ is extremized:

$$|E|^2 \propto \cos^2(-\theta) + \cos^2(-\theta + \alpha)$$

$$\frac{d|E|^2}{d\theta} = 2\cos(-\theta)\sin(-\theta) + 2\cos(-\theta+\alpha)\sin(-\theta+\alpha) = 0.$$

$$\sin(-2\theta) = -\sin(-2\theta + 2\alpha)$$

$$\sin(-2\theta) = \sin(2\theta - 2\alpha)$$

$$-2\theta = 2\theta - 2\alpha, \quad \boxed{\theta = \frac{\alpha}{2}}$$

$$|E|^2\Big|_{\theta=\frac{\alpha}{2}} = \frac{25}{2}\left[\cos^2\left(-\frac{\alpha}{2}\right) + \cos^2\left(-\frac{\alpha}{2}+\alpha\right)\right]$$

$$= \frac{25}{2}\left[2\cos^2\left(\frac{\alpha}{2}\right)\right]$$

$$\approx \frac{25}{2}\left[2\cos^2(0.14)\right]$$

$$= 25\cos^2(0.14) \approx \boxed{24.513.}$$

$\Rightarrow$ The semimajor axis is $\sqrt{24.513} = 4.95$,

the semiminor axis is $\sqrt{25-24.513} = 0.698$

Elliptical Polarization basis.

$$s_0 + s_3 = 2a_+^2 , \qquad s_0 - s_3 = 2a_-^2$$

$$\Rightarrow \quad a_+ = 4 , \qquad a_- = 3$$

$$\delta_- - \delta_+ = \cos^{-1}\left[\frac{s_1}{2a_1 a_2}\right] = \frac{\pi}{2}$$

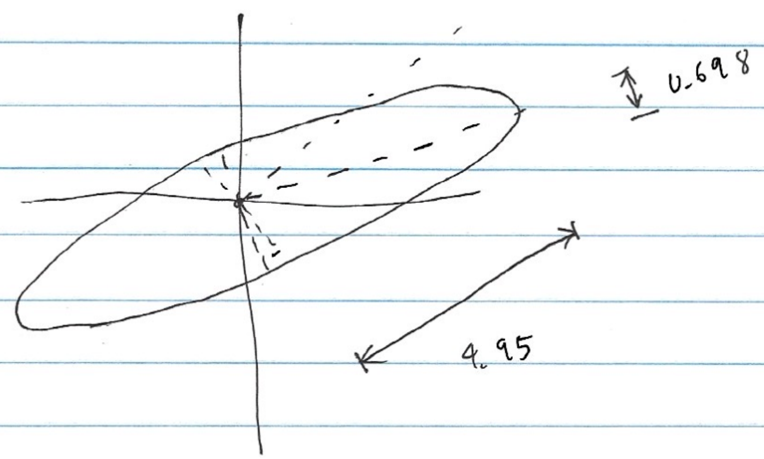
WLOG, $\delta_+ = 0$, $\delta_- = \frac{\pi}{2}$, $E_+ = 4$, $E_- = 3e^{i\pi/2}$.

$$\Rightarrow \vec{E}(\vec{x},t) = \mathrm{Re}\left[4\vec{\varepsilon}_+ + 3e^{i\frac{\pi}{2}}\vec{e}_-\right]e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

~~For fixed $\vec{x}$,~~ $\cancel{E_x}$

Solving for $\vec{E}$ ratio of semimajor axis to semiminor axis gives $\frac{1+3/4}{1-3/4} = 7$.

$\Rightarrow$ Semimajor axis has length. 4.95,
semiminor axis has length. 0.698.



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